The Theory of Integer Partitions, II

Trine Mathematics Colloquium

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2021 September 23

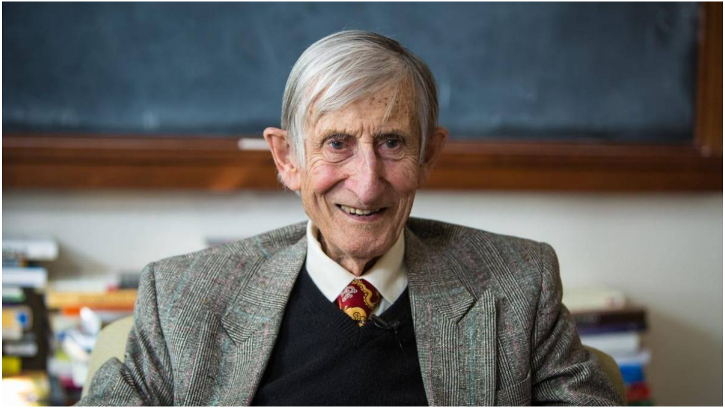
Agenda

- Recap
- New Tools
  - Young Tableaux
  - Two-Variable Generating Functions
  - Frobenius Symbols
- Results

A **partition** λ is a finite non-increasing sequence of positive integers. The **weight** of λ is the sum of its integer parts.

$$(4,3,2,1) \qquad (1,1,1) \qquad () \qquad (1000,999,17,5)$$

The relation $\lambda \vdash n$ means "the weight of λ is n". We also say that "λ is a partition of n".



Freeman Dyson (1923 - 2020) was interested in finding a simpler proof of the Ramanujan congruences by measuring statistics of individual partitions.

Let λ be a partition. The **rank** of λ is an integer equal to the largest part of λ minus the number of parts which occur in λ.

$$r(4, 2, 1) = 4 - 3 = 1.$$

As conjectured by Dyson, grouping the partitions according to their rank gives a combinatoric proof of the first two Ramanujan Congruences,

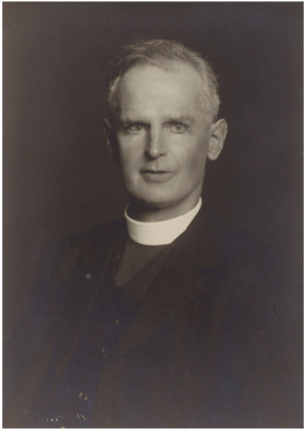
$$p(5n + 4) \equiv 0 \pmod{5}$$
$$p(7n + 5) \equiv 0 \pmod{7}.$$

Here are the partitions $\lambda \vdash 4$.

λ	$r(\lambda)$	$r(\lambda) \mod 5$
(4)	3	3
$(3,1)$	1	1
$(2,2)$	0	0
$(2,1,1)$	-1	4
$(1,1,1,1)$	-3	2

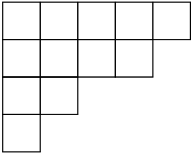
Each of the residues 0, 1, 2, 3, and 4 occur the same number of times in the last column.

You might also notice something interesting in the middle column.



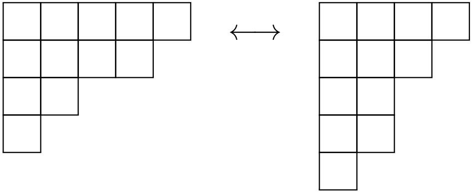
Alfred Young (1873 - 1940) studied partitions as they relate to modern algebra and graph theory.

Let $\lambda = (n_1, n_2, \ldots, n_k)$ be a partition. The **Young tableau** of λ is an array of boxes, where the ith row consists of n_i boxes.



Pictured above is the tableau for $\lambda = (5, 4, 2, 1)$.

The **conjugate** partition to λ is obtained by mirroring the Young tableau of λ across its NE-SW diagonal. This gives an involution on the set of partitions $\lambda \vdash n$.



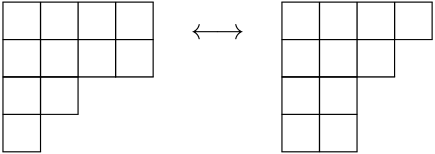
Here, $\lambda = (5, 4, 2, 1)$ and $\overline{\lambda} = (4, 3, 2, 2, 1)$.

Proofs Without Words

Theorem

Let $n \geq 0$ and $m \in \mathbb{Z}$. The number of partitions $\lambda \vdash n$ with $r(\lambda) = m$ is equal to the number of partitions $\mu \vdash n$ with $r(\mu) = -m$.

Proof



$$\lambda \quad \longleftrightarrow \quad \mu$$

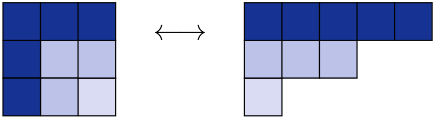
$\square$

Proofs Without Words

Theorem

Let $n \geq 0$. The number of self conjugate partitions $\lambda \vdash n$ is equal to the number of partitions $\mu \vdash n$ consisting of disctinct odd parts.

Proof



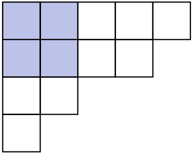
$$\lambda \longleftrightarrow \mu$$

$\square$


WILLIAM PITT DURFEE

William Pitt Durfee (1855 - 1941) was an American mathematician, and a dean at Hobart College in Geneva, New York.

Every partition λ has a **Durfee square**, which is the largest square that fits in the upper left corner of its Young tableau.



Here, the Durfee square has side length 2.

(If you write academic papers, this is h-index for partitions.)

We can use Durfee squares to establish a two-variable generating function for the ranks of partitions.

Lemma

Let P denote the set of all partitions, and let $|\lambda|$ denote the weight of the partition λ. Then,

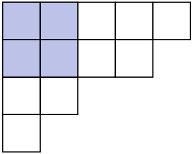
$$\sum_{\lambda \in P} z^{r(\lambda)} q^{|\lambda|} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(zq;q)_n (q/z;q)_n}.$$

Here, $(a;q)_n$ stands for the q**-rising factorial**,

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Proof

Fix $n \geq 0$. Any partition λ whose Durfee square has side length n can be disected into three parts:



First, a square with side length n.
Then, at most n rows of arbitrary size to the right of the square.
Last, at most n columns of arbitrary size below the square.

The generating function for Young diagrams with at most n rows is

$$\frac{1}{(zq;q)_n} = (1+zq^1+z^2q^{1+1}+\cdots)\times\cdots\times(1+zq^n+z^2q^{n+n}+\cdots).$$

Here, the exponent of z counts the total number of columns, and the exponent of q in the ith factor counts the number of columns of height i.

By symmetry, $1/(q/z;q)_n$ is generating function for Young diagrams with at most n columns, where the exponent of z^{-1} counts the total number of rows.

Taking the product of generating functions is equivalent to taking a Cartesian product of sets. Therefore, the generating function for partitions whose Durfee square has side length n is given by

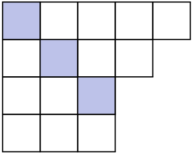
$$\frac{q^{n^2}}{(zq;q)_n(q/z;q)_n},$$

or 1 in the case $n = 0$. The rank of any such tableau is its number of columns minus its number of rows.

Summing over all $n \geq 0$ produces the result. $\square$

Ferdinand Georg Frobenius (1849 - 1917) established deep relationships between partitions and the symmetric groups in modern algebra.

The **Frobenius symbol** is a $2 \times k$ array which enumerates the number of boxes to the right of the main diagonal of a Young diagram, and then the number of boxes below the main diagonal.



$$(5,4,3,3) \leftrightarrow \begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

The weight of a Frobenius symbol with n columns is equal to the sum of its entries plus n, because the length of the diagonal is implied by the length of the array.

Lemma

The generating function for the set of Frobenius symbols is given by the coefficient of z^0 *in*

$$(-zq;q)_\infty(-1/z;q)_\infty = \left[(1+(zq)q^0)\times(1+(zq)q^1)\times(1+(zq)q^2)\times\cdots\right] \times\left[(1+q^0/z)\times(1+q^1/z)\times(1+q^2/z)\times\cdots\right].$$

Proof Sketch

This series generates pairs of partitions into distinct parts. The products which yield z^0 must generate the same number of parts in the top and bottom rows of the Frobenius representation. $\square$

George Andrews (1938 -) has been a leading figure in the theory of partitions through the 20th century and continues to publish today.

Frank Garvan (1955 -) is another leading figure in the theory of partitions. He maintains a software suite for computer calculation of generating functions at `qseries.org`.

Let λ be a partition. If λ does not contain any parts of size 1, then the **crank** of λ is defined to be equal to the largest part of λ.

Otherwise, define $w(\lambda)$ to be the number of parts of size 1 which occur in λ. Then define $m(\lambda)$ to be the number of parts of λ which are greater than $w(\lambda)$. Finally, we define $c(\lambda) = m(\lambda) - w(\lambda)$.

$$c(5,4,2) = 5$$
$$c(5,4,1) = \#\{5,4\} - 1 = 1$$

Lemma (Andrews, Garvan)

The two-variable generating function for the cranks of partitons is given by

$$\sum_{\lambda \in P} z^{c(\lambda)} q^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1-q^i}{(1-zq^i)(1-q^i/z)} = \frac{(q;q)_\infty}{(zq;q)_\infty (q/z;q)_\infty}.$$

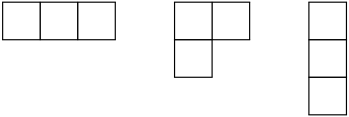
We observe that the crank function exhibits the same kind of symmetry as does the rank function. Whenever $c(\lambda) = m$ there is a corresponding $c(\mu) = -m$.

Theorem (Andrews, Dastidar, M.; 2021)

For each $j \geq 0$, the number of partitions $\lambda \vdash n$ with crank $c(\lambda) > j$ equals one half of the number of j's in all Frobenius symbols for the partitions of n.

$$c(3) = 3$$
$$c(2,1) = 0$$
$$c(1,1,1) = -3$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



Proof Sketch

The generating function for the partitions $\lambda \vdash n$ with crank $c(\lambda) > j$ is given by

$$(q;q)_\infty \sum_{m=0}^{\infty} \frac{z^{-m} q^m}{(q;q)_m} \sum_{n=m+j+1}^{\infty} \frac{z^n q^n}{(q;q)_n}.$$

The generating function for the number of j's occuring in all Frobenius symbols for the partitions of n is given by the coefficient of z^0 in

$$\left\{ \frac{\partial}{\partial y} \frac{(1+yzq^{j+1})(1+yq^j/z)}{(1+zq^{j+1})(1+q^j/z)} (-zq;q)_\infty (-1/z;q)_\infty \right\} \Bigg|_{y=1}.$$

The first series reduces to

$$\frac{1}{(q;q)_\infty}\sum_{m=1}^{\infty}(-1)^{m-1}q^{\binom{m+1}{2}+mj}.$$

The second series reduces to

$$\frac{2}{(q;q)_\infty}\sum_{m=1}^{\infty}(-1)^{m-1}q^{\binom{m+1}{2}+mj}.$$

Thus, each of the coefficients of the first series are one-half the corresponding coefficient of the second series. $\square$

Theorem (Andrews, Dastidar, M.; 2021)

Let λ be a partition of n with $c(\lambda) = k > 0$. Then there is a one-to-one correspondence between λ and a set consisting of two occurrences of each of the integers i with $0 \leq i \leq k-1$ among all of the parts of the Frobenius symbols for the partitions of n.

Corollary

For all $n \geq 0$, the number of entries in all Frobenius symbols for partitions of n is equal to

$$\sum_{\substack{\lambda \vdash n \\ c(\lambda) > 0}} 2c(\lambda).$$

For $n \geq 0$, the **second crank moment** is defined to be

$$M_2(n) = \sum_{\lambda \vdash n} c(\lambda)^2.$$

Theorem (Dyson; 1989)

Let $p(n)$ denote the number of partitions of n. For all $n \geq 0$,

$$\frac{1}{2} M_2(n) = np(n).$$

This result has been given many different alternate proofs since its initial discovery.

Proof

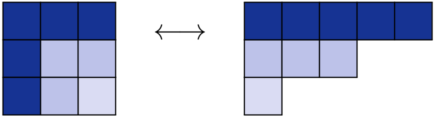
By symmetry,

$$M_2(n) = \sum_{\lambda \vdash n} c(\lambda)^2 = 2 \sum_{\substack{\lambda \vdash n \\ c(\lambda)=k>0}} k^2.$$

That is, $\frac{1}{2}M_2(n)$ may be computed by summing over partitions with positive crank only.

This is exactly the domain of our main result.

For any positive integer k, we have that k^2 is equal to $1 + 3 + \cdots + (2(k-1)+1)$.



Then,

$$\frac{1}{2}M_2(n) = \sum_{\substack{\lambda \vdash n \\ c(\lambda)=k>0}} k^2 = \sum_{\substack{\lambda \vdash n \\ c(\lambda)=k>0}} \sum_{i=0}^{k-1} (2i+1).$$

Each such λ corresponds to a set of parts

$$\{0, 0, \ldots, i, i, \ldots, (k-1), (k-1)\}$$

taken from all Frobenius symbols of partitions of n. The weight associated to each i is then $2i+1$, accounting for the diagonal of the Young tableau.

Each of the $p(n)$ Frobenius symbols has weight n. Therefore,

$$\sum_{\substack{\lambda \vdash n \\ c(\lambda)=k>0}} \sum_{i=0}^{k-1} (2i+1) = np(n).$$

$\square$

Conclusion

I have yet to discuss more intricate aspects of partition theory, such as Euler's Pentagonal Number Theorem, the Rogers-Ramanujan identities, and the Jacobi triple product. Please join me again in the future when I discuss more of these fascinating objects.

Thank You!